A brief introduction to pre-Lie algebras

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Outline

- What is a pre-Lie algebra?
- Examples of pre-Lie algebras
- Pre-Lie algebras and classical Yang-Baxter equation
- Pre-Lie algebras and vertex (operator) algebras
What is a pre-Lie algebra?

**Definition**

A *pre-Lie algebra* $A$ is a vector space with a binary operation $(x, y) \rightarrow xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \quad \forall x, y, z \in A. \quad (1)$$

- Other names:
  1. left-symmetric algebra
  2. Koszul-Vinberg algebra
  3. quasi-associative algebra
  4. right-symmetric algebra
  5. ...
What is a pre-Lie algebra?

Proposition

Let $A$ be a pre-Lie algebra.

1. **The commutator**

   \[
   [x, y] = xy - yx, \quad \forall x, y \in A
   \]  

   defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of $A$.

2. For any $x, y \in A$, let $L(x)$ denote the left multiplication operator. Then Eq. (1) is just

   \[
   [L(x), L(y)] = L([x, y]), \quad \forall x, y \in A,
   \]

   which means that $L : \mathfrak{g}(A) \to gl(\mathfrak{g}(A))$ with $x \mapsto L(x)$ gives a representation of the Lie algebra $\mathfrak{g}(A)$. 

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What is a pre-Lie algebra?

- The “left-symmetry” of associators.
  That is, for any $x, y, z \in A$, the associator
  \[
  (x, y, z) = (xy)z - x(yz)
  \] (4)
  satisfies
  \[
  (x, y, z) = (y, x, z)
  \] (5)
  which is equivalent to Eq. (1).
What is a pre-Lie algebra?

- A geometric interpretation of “left-symmetry”:
  Let $G$ be a Lie group with a left-invariant affine structure:
  there is a flat torsion-free left-invariant affine connection $\nabla$ on $G$,
  namely, for all left-invariant vector fields $X, Y, Z \in \mathfrak{g} = T(G),$

  $$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z = 0,$$
  \[ \tag{6} \]

  $$T(X, Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0.$$
  \[ \tag{7} \]

  This means both the curvature $R(X, Y)$ and torsion $T(X, Y)$ are zero for the connection $\nabla$. If we define

  $$\nabla_X Y = XY,$$
  \[ \tag{8} \]

  then the identity (1) for a pre-Lie algebra exactly amounts to Eq. (6).
What is a pre-Lie algebra?

- The origins and roles of pre-Lie algebras:
  - Cohomology and deformations of associative algebras: Gerstenhaber (1964), · · ·
  - Symplectic and Kähler structures on Lie groups: Chu (1973), Shima (1980), Dardie and Medina (1995-6), · · ·
  - Complex structures on Lie groups: Andrada and Salamon (2003), · · ·
  - Rooted trees: Cayley (1896), · · ·
  - Combinatorics: K. Ebrahimi-Fard (2002), · · ·
  - Operads: Chapoton and Livernet (2001), · · ·
  - Vertex algebras: Bakalov and Kac (2002), · · ·
What is a pre-Lie algebra?

- Phase spaces: Kuperschmidt(1994), · · ·
- Integrable systems: Bordemann(1990), Winterhalder(1997), · · ·
- Classical and quantum Yang-Baxter equations: Svinolupov and Sokolov(1994), Etingof and Soloviev(1999), Golubschik and Sokolov(2000), · · ·
- Poisson brackets and infinite-dimensional Lie algebras: Gel’fand and Dorfman(1979), Dubrovin and Novikov(1984), Balinskii and Novikov(1985), · · ·
- Quantum field theory and noncommutative geometry: Connes and Kreimer(1998), · · ·
What is a pre-Lie algebra?

- Interesting structures:
  Let $A$ be a pre-Lie algebra and $\mathfrak{g}(A)$ be the sub-adjacent Lie algebra.

  - On $A \oplus A$ as a direct sum of vector spaces:
    1. $\mathfrak{g}(A) \ltimes_{\text{ad}} \mathfrak{g}(A) \Rightarrow$
    2. $\mathfrak{g}(A) \ltimes_{L} \mathfrak{g}(A) \Rightarrow$ Complex structures

  - On $A \oplus A^*$ as a direct sum of vector spaces ($A^*$ is the dual space of $A$):
    1. $\mathfrak{g}(A) \ltimes_{\text{ad}^*} \mathfrak{g}(A)^* \Rightarrow$ Manin Triple (Lie bialgebra)
    2. $\mathfrak{g}(A) \ltimes_{L^*} \mathfrak{g}(A)^* \Rightarrow$ Symplectic structures
What is a pre-Lie algebra?

- Main Problems: Non-associativity!
  1. There is not a suitable (and computable) representation theory.
  2. There is not a complete (and good) structure theory.

Example: There exists a transitive simple pre-Lie algebra which combines “simplicity” and certain “nilpotence”:

\[ e_1 e_2 = e_2, \quad e_1 e_3 = -e_3, \quad e_2 e_3 = e_3 e_2 = e_1. \]

- Ideas: Try to find more examples!
  1. The relations with other topics (including application).
  2. Realize by some known structures.
Examples of pre-Lie algebras

- Construction from commutative associative algebras

**Proposition**

(S. Gel’fand) Let \((A, \cdot)\) be a commutative associative algebra, and \(D\) be its derivation. Then the new product

\[
a \ast b = a \cdot Db, \quad \forall a, b \in A
\]

makes \((A, \ast)\) become a pre-Lie algebra.

In fact, \((A, \ast)\) is a Novikov algebra which is a pre-Lie algebra satisfying an additional condition \(R(a)R(b) = R(b)R(a)\), where \(R(a)\) is the right multiplication operator for any \(a \in A\).

\(R(a)R(b) = R(b)R(a)\) defines a NAP algebra.
Examples of pre-Lie algebras

- Background: Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus.

\[
\{u(x_1), v(x_2)\} = \frac{\partial}{\partial x_1}((uv)(x_1))x_1^{-1}\delta\left(\frac{x_1}{x_2}\right) \\
+ (uv + vu)(x_1)\frac{\partial}{\partial x_1}x_1^{-1}\delta\left(\frac{x_1}{x_2}\right).
\]

- Generalization: (Bai-Meng’s conjecture) Every Novikov algebra can be realized as the algebras (9) and their (compatible) linear deformations.

- Conclusion:
  1. (Bai-Meng) The conjecture holds in dimension \( \leq 3 \).
  2. (Dzhumadil’dav-Lofwall) Any Novikov algebra is a quotient of a subalgebra of an algebra given by Eq. (9).
Examples of pre-Lie algebras

- **Construction from linear functions**

**Proposition**

Let $V$ be a vector space over the complex field $\mathbb{C}$ with the ordinary scalar product $(,)$ and $a$ be a fixed vector in $V$, then

$$u \ast v = (u, v)a + (u, a)v, \forall u, v \in V,$$

defines a pre-Lie algebra on $V$.

- **Background:** The integrable (generalized) Burgers equation

$$U_t = U_{xx} + 2U \ast U_x + (U \ast (U \ast U)) - ((U \ast U) \ast U).$$

- **Generalization:** Pre-Lie algebras from linear functions.

- **Conclusion:** The pre-Lie algebra given by Eq. (11) is simple.
Examples of pre-Lie algebras

- **Construction from associative algebras**

**Proposition**

Let $(A, \cdot)$ be an associative algebra and $R : A \to A$ be a linear map satisfying

$$R(x) \cdot R(y) + R(x \cdot y) = R(R(x) \cdot y + x \cdot R(y)), \forall x, y \in A. \quad (13)$$

Then

$$x \ast y = R(x) \cdot y - y \cdot R(x) - x \cdot y, \quad \forall x, y \in A \quad (14)$$

defines a pre-Lie algebra.

- The above $R$ is called *Rota-Baxter map of weight 1*.
- Generalization: Approach from associative algebras.
- Related to the so-called “modified classical Yang-Baxter equation”.

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Examples of pre-Lie algebras

- **Construction from Lie algebras (I)**

1. **Question 1**: Whether there is a compatible pre-Lie algebra on every Lie algebra?
   - **Answer**: No!
   - Necessary condition: The sub-adjacent Lie algebra of a finite-dimensional pre-Lie algebra $A$ over an algebraically closed field with characteristic 0 satisfies
     
     $[\mathfrak{g}(A), \mathfrak{g}(A)] \neq \mathfrak{g}(A).$  

     $(15)$

     $\implies \mathfrak{g}(A)$ can not be semisimple.

2. **Question 2**: How to construct a compatible pre-Lie algebra on a Lie algebra?
   - **Answer**: etale affine representation $\iff$ bijective 1-cocycle.

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Examples of pre-Lie algebras

Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to gl(V)$ be a representation of $\mathfrak{g}$. A 1-cocycle $q$ associated to $\rho$ (denoted by $(\rho, q)$) is defined as a linear map from $\mathfrak{g}$ to $V$ satisfying

$$q[x, y] = \rho(x)q(y) - \rho(y)q(x), \forall x, y \in \mathfrak{g}. \quad (16)$$

Let $(\rho, q)$ be a bijective 1-cocycle of $\mathfrak{g}$, then

$$x \ast y = q^{-1} \rho(x)q(y), \forall x, y \in A, \quad (17)$$

defines a pre-Lie algebra structure on $\mathfrak{g}$. Conversely, for a pre-Lie algebra $A$, $(L, id)$ is a bijective 1-cocycle of $\mathfrak{g}(A)$.

Proposition

There is a compatible pre-Lie algebra on a Lie algebra $\mathfrak{g}$ if and only if there exists a bijective 1-cocycle of $\mathfrak{g}$. 

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• Application: Providing a linearization procedure of classification.
• Conclusion: Classification of complex pre-Lie algebras up to dimension 3.
• Generalization: classical r-matrices! $\implies$ a realization by Lie algebras
Pre-Lie algebras and classical Yang-Baxter equation

Definition

Let $\mathfrak{g}$ be a Lie algebra and $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$. $r$ is called a solution of **classical Yang-Baxter equation (CYBE)** in $\mathfrak{g}$ if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \text{ in } U(\mathfrak{g}),$$  \hspace{1cm} (18)

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1; \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i; \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \hspace{1cm} (19)$$

$r$ is said to be **skew-symmetric** if

$$r = \sum_i (a_i \otimes b_i - b_i \otimes a_i).$$  \hspace{1cm} (20)

We also denote $r^{21} = \sum_i b_i \otimes a_i$. 
Pre-Lie algebras and classical Yang-Baxter equation

Definition

Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to gl(V)$ be a representation of $\mathfrak{g}$. A linear map $T : V \to \mathfrak{g}$ is called an $O$-operator if $T$ satisfies

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \forall u, v \in V. \quad (21)$$
Relationship between $\mathcal{O}$-operators and CYBE

From CYBE to $\mathcal{O}$-operators

**Proposition**

Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Suppose that $r$ is skew-symmetric. Then $r$ is a solution of CYBE in $\mathfrak{g}$ if and only if $r$ regarded as a linear map from $\mathfrak{g}^* \to \mathfrak{g}$ is an $\mathcal{O}$-operator associated to $\text{ad}^*$. 

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Notation: let $\rho : g \rightarrow gl(V)$ be a representation of the Lie algebra $g$. On the vector space $g \oplus V$, there is a natural Lie algebra structure (denoted by $g \ltimes_{\rho} V$) given as follows:

$$\left[x_1 + v_1, x_2 + v_2\right] = \left[x_1, x_2\right] + \rho(x_1)v_2 - \rho(x_2)v_1,$$

(22)

for any $x_1, x_2 \in g, v_1, v_2 \in V$.

**Proposition**

Let $g$ be a Lie algebra. Let $\rho : g \rightarrow gl(V)$ be a representation of $g$ and $\rho^* : g \rightarrow gl(V^*)$ be the dual representation. Let $T : V \rightarrow g$ be a linear map which is identified as an element in $g \otimes V^* \subset (g \ltimes_{\rho^*} V^*) \otimes (g \ltimes_{\rho^*} V^*)$. Then $r = T - T^{21}$ is a skew-symmetric solution of CYBE in $g \ltimes_{\rho^*} V^*$ if and only if $T$ is an $\mathcal{O}$-operator.
○ Relationship between $\mathcal{O}$-operators and pre-Lie algebras

**Proposition**

Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to gl(V)$ be a representation. Let $T : V \to \mathfrak{g}$ be an $\mathcal{O}$-operator associated to $\rho$, then

$$u \ast v = \rho(T(u))v, \quad \forall u, v \in V$$

(23)

defines a pre-Lie algebra on $V$. 
Lemma

Let $\mathfrak{g}$ be a Lie algebra and $(\rho, V)$ is a representation. Suppose $f : \mathfrak{g} \rightarrow V$ is invertible. Then $f$ is a 1-cocycle of $\mathfrak{g}$ associated to $\rho$ if and only if $f^{-1}$ is an $\mathcal{O}$-operator.

Corollary

Let $\mathfrak{g}$ be a Lie algebra. There is a compatible pre-Lie algebra structure on $\mathfrak{g}$ if and only if there exists an invertible $\mathcal{O}$-operator of $\mathfrak{g}$. 
○ Relationship between pre-Lie algebras and CYBE

From pre-Lie algebras to CYBE

Proposition

Let $A$ be a pre-Lie algebra. Then

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i)$$

(24)

is a solution of the classical Yang-Baxter equation in the Lie algebra $\mathfrak{g}(A) \ltimes_{L^*} \mathfrak{g}(A)^*$, where $\{e_1, ..., e_n\}$ is a basis of $A$ and $\{e_1^*, ..., e_n^*\}$ is the dual basis.
From CYBE to pre-Lie algebras (Construction from Lie algebras (II))

**Lemma**

Let $\mathfrak{g}$ be a Lie algebra and $f$ be a linear transformation on $\mathfrak{g}$. Then on $\mathfrak{g}$ the new product

$$x \ast y = [f(x), y], \forall x, y \in \mathfrak{g}$$

(25)

defines a pre-Lie algebra if and only if

$$[f(x), f(y)] - f([f(x), y] + [x, f(y)]) \in C(\mathfrak{g}), \forall x, y \in \mathfrak{g},$$

(26)

where $C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0, \forall y \in \mathfrak{g}\}$ is the center of $\mathfrak{g}$. 
Corollary

Let $\mathfrak{g}$ be a Lie algebra. Let $r : \mathfrak{g} \to \mathfrak{g}$ be an $O$-operator associated to $\text{ad}$, that is,

$$[r(x), r(y)] - r([r(x), y] + [x, r(y)]) = 0, \quad \forall x, y \in \mathfrak{g}. \quad (27)$$

Then Eq. (25) defines a pre-Lie algebra.

Remark

1. Eq. (27) is called operator form of CYBE;
2. Eq. (27) is the Rota-Baxter operator of weight zero in the context of Lie algebras.
3. If there is a nondegenerate invariant bilinear form on $\mathfrak{g}$ (that is, as representations, $\text{ad}$ is isomorphic to $\text{ad}^*$) and $r$ is skew-symmetric, then $r$ satisfies CYBE if and only if $r$ regarded as a linear transformation on $\mathfrak{g}$ satisfies Eq. (27).
An algebraic interpretation of “left-symmetry”:

Let \( \{e_i\} \) be a basis of a Lie algebra \( \mathfrak{g} \). Let \( r : \mathfrak{g} \to \mathfrak{g} \) be an \( \mathcal{O} \)-operator associated to \( \text{ad} \). Set \( r(e_i) = \sum_{j \in I} r_{ij} e_j \). Then the basis-interpretation of Eq. (25) is given as

\[
e_i \ast e_j = \sum_{l \in I} r_{il} [e_l, e_j].
\]

(28)

Such a construction of left-symmetric algebras can be regarded as a Lie algebra “left-twisted” by a classical \( r \)-matrix.

On the other hand, let us consider the right-symmetry. We set

\[
e_i \cdot e_j = [e_i, r(e_j)] = \sum_{l \in I} r_{jl} [e_i, e_l].
\]

(29)

Then the above product defines a right-symmetric algebra on \( \mathfrak{g} \), which can be regarded as a Lie algebra “right-twisted” by a classical \( r \)-matrix.

Application: Realization of pre-Lie algebras by Lie algebras.
A **vertex algebra** is a vector space $V$ equipped with a linear map

$$Y : V \rightarrow \text{Hom}(V, V((x))), \; v \rightarrow Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

and equipped with a distinguished vector $1 \in V$ such that

$$Y(1, x) = 1;$$

$$Y(v, x)1 \in V[[x]] \quad \text{and} \quad Y(v, x)1|_{x=0}(= v_{-1}1) = v, \; \forall v \in V,$$

and for $u, v \in V$, there is Jacobi identity:

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1)$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2).$$
Pre-Lie algebras and vertex (operator) algebras

Proposition

Let \((V, Y, 1)\) be a vertex algebra. Then

\[
a * b = a_{-1} b
\]  

(32)

defines a pre-Lie algebra.

That is,

\[
(a_{-1} b)_{-1} c - a_{-1} (b_{-1} c) = (b_{-1} a)_{-1} c - b_{-1} (a_{-1} c).
\]  

(33)

In fact, by Borcherd’s identities:

\[
(a_m(b))_n(c) = \sum_{i \geq 0} (-1)^i C^i_m ((a_{m-i}(b_{n+i}(c)) - (-1)^m b_{m+n-i}(a_i(c))),
\]  

(34)

let \(m = n = -1\), we have

\[
(a_{-1} b)_{-1} c - a_{-1} (b_{-1} c) = \sum_{i \geq 0} (a_{-2-i}(b_i c) + b_{-2-i}(a_i c)).
\]  

(35)
Pre-Lie algebras and vertex (operator) algebras

Proposition

A vertex algebra is equivalent to a pre-Lie algebra and an algebra names Lie conformal algebra with some compatible conditions.
A Lie conformal algebra is a $\mathbb{C}[D]$-module $V$ endowed a $\mathbb{C}$-linear map $V \otimes V \to \mathbb{C}[\lambda] \otimes V$ denoted by $a \otimes b \mapsto [a_{\lambda}b]$, satisfying

$$[(Da)_{\lambda}b] = -\lambda[a_{\lambda}b];$$

$$[a_{\lambda}(Db)] = (\lambda + D)[a_{\lambda}b];$$

$$[b_{\lambda}a] = [a_{-\lambda - T}b];$$

$$[a_{\lambda}b]_{\lambda + \mu}c = [a_{\lambda}[b_{\mu}c] - [b_{\mu}[a_{\lambda}c]]].$$

Let $(V, Y, 1)$ be a vertex algebra. Then

$$a_{\lambda}b = \text{Res}_x e^{\lambda x} Y(a, x)b = \sum_{n \geq 0, \text{finite}} \lambda^n a_n b / n!. \quad (36)$$
Proposition

A vertex algebra is a pair \((V, 1)\), where \(V\) is a \(\mathbb{C}[D]\)-module and \(1 \in V\), endowed two operations:

1. \(V \otimes V \rightarrow \mathbb{C}[\lambda] \otimes V, a \otimes b \rightarrow [a_\lambda b],\) Lie conformal algebra;
2. \(V \otimes V \rightarrow V, a \otimes b \rightarrow ab,\) a differential algebra with unit 1 and derivation \(D\).

They satisfy the following conditions:

1. \((ab)c - a(bc) = (\int_0^D d\lambda a)[b_\lambda c] + (\int_0^D d\lambda b)[a_\lambda c];\)
2. \(ab - ba = \int_{-D}^0 d\lambda a[a_\lambda b];\)
3. \([a_\lambda (bc)] = [a_\lambda b]c + b[a_\lambda c] + \int_0^\lambda d\mu[[a_\lambda b] - \mu c].\)
Roughly speaking,

\[ Y(a, x)b = Y(a, x)_+ b + Y(a, x)_- b, \]

where

\[ Y(a, x)_+ b = (e^{xD}a)_- 1 b, \]
\[ Y(a, x)_- b = (a_ - \partial_x b)(x^{-1}). \]
Novikov algebras and infinite-dimensional Lie algebras and vertex (operator) algebras

**Proposition**

Let $A$ be a finite-dimensional algebra with a bilinear product $(a, b) \rightarrow ab$. Set

$$A = A \otimes \mathbb{C}[t, t^{-1}]. \quad (37)$$

Then the bracket

$$[a \otimes t^m, b \otimes t^n] = (-mab + nba) \otimes t^{m+n-1}, \quad \forall a, b \in A, \ m, n \in \mathbb{Z} \quad (38)$$

defines a Lie algebra structure on $A$ if and only if $A$ is a Novikov algebra with the product $ab$. 
Let $A$ be a Novikov algebra. There is a $\mathbb{Z}$–grading on the Lie algebra $A$ defined by Eqs. (37) and (38):

$$A = \bigoplus_{n \in \mathbb{Z}} A(n), \quad A(n) = A \otimes t^{-n+1}.$$ 

Moreover,

$$A_{(n \leq 1)} = \bigoplus_{n \leq 1} A(n) = \bigoplus_{n \leq 0} A(n) \oplus A(1)$$

is a Lie subalgebra of $A$. For any $a \in A$, we define the generating function:

$$a(x) = \sum_{n \in \mathbb{Z}} a_n x^{-n-1} = \sum_{n \in \mathbb{Z}} (a \otimes t^n) x^{-n-1} \in A[[x, x^{-1}]].$$

Then we have

$$[a(x_1), b(x_2)] = (ab+ba)(x_1) \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) + \left[ \frac{\partial}{\partial x_1} ab(x_1) \right] x_2^{-1} \delta \left( \frac{x_1}{x_2} \right).$$
Let $C$ be the trivial module of $A_{(n \leq 1)}$ and we can get the following (Verma) module of $A$:

$$\hat{A} = U(A) \otimes U(A_{(n \leq 1)}) C,$$

where $U(A)$ ($U(A_{(n \leq 1)})$) is the universal enveloping algebra of $A$ ($A_{(n \leq 1)}$). $\hat{A}$ is a natural $\mathbb{Z}$-graded $A$-module:

$$\hat{A} = \bigoplus_{n \geq 0} \hat{A}(n),$$

where

$$\hat{A}(n) = \{ a^{(1)}_{-m_1} \cdots a^{(r)}_{-m_r} \cdot 1 | m_1 + \cdots + m_r = n-r, \ m_1 \geq \cdots \geq m_r \geq 1, \ r \geq 0 \}$$
Let $A$ be a Novikov algebra. Then there exists a unique vertex algebra structure $(\hat{A}, Y, 1)$ on $\hat{A}$ such that $1 = 1 \in \mathbb{C}$ and $Y(a, x) = a(x), \ \forall a \in A$, and

$$Y(a^{(1)}_{(n_1)} \cdots a^{(r)}_{(n_r)} 1, x) = a^{(1)}(x)^{n_1} \cdots a^{(r)}(x)^{n_r} 1_{\hat{A}}$$

(39)

for any $r \geq 0, a^{(i)} \in A, n_i \in \mathbb{Z}$, where

$$a(x)_n b(x) = \text{Res}_{x_1} ((x_1 - x)^n a(x_1)b(x) - (-x + x_1)^n b(x)a(x_1)).$$

(40)
Pre-Lie algebras and vertex (operator) algebras

Theorem

Let \((V, Y, 1)\) be a vertex algebra with the following properties:

1. \(V = \bigoplus_{n \geq 0} V(n), \ V(0) = C1, \ V(1) = 0;\)
2. \(V\) is generated by \(V(2)\) with the following property
   \[
   V = \text{span}\{a^{(1)}_{-m_1} \cdots a^{(r)}_{-m_r} 1 | m_i \geq 1, r \geq 0, a^{(i)} \in V(2)\}
   \]
3. \(\text{Ker}D = C,\) where \(D\) is a linear transformation of \(V\) given by
   \(D(v) = v_{-2}1, \forall v \in V.\)
4. \(V\) is a graded (by the integers) vertex algebra, that is,
   \(1 \in V(0)\) and \(V(i)_n V(j) \subset V(i+j-n-1).\)
Theorem

(Continued) Then $V$ is generated by $V_{(2)}$ with the following property

$$V = \text{span}\{a_{-m_1}^{(1)} \cdots a_{-m_r}^{(r)} 1 | m_1 \geq \cdots \geq m_r \geq 1, r \geq 0, a^{(i)} \in V_{(2)}\}$$

and $V_{(2)}$ is a Novikov algebra with a product $(a, b) \rightarrow a \ast b$ given by

$$a \ast b = -D^{-1}(b_0 a).$$
Furthermore, if the Novikov algebra $A$ has an identity $e$, then $-e$ corresponds to the Virasoro element which gives a vertex operator algebra structure with zero central charge. With a suitable central extension, the non-zero central charge will lead to a commutative associative algebra with a nondegenerate symmetric invariant bilinear form (the so-called Frobenius algebra).

**Corollary**

For the above vertex algebra $(V, Y, 1)$, the algebra given by

$$a \ast b = a_{-1}b$$

is a graded pre-Lie algebra, that is $V_m \ast V_n \subset V_{m+n}$. 
Thank You!